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A PARTICULAR SOLUTION OF THE PRANDTL-REUSS EQUATION

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The complete dynamic equations of Prandtl-Reuss [1] are examined in the rectangular region. An exact solution is given for a problem which corresponds to some specially selected boundary conditions and initial conditions.

The obtained solution is used to evaluate the correctness of some assumptions which are applicable in the approximate solution of these equations [2].

1. The equations of Prandtl-Reuss are used for the description of dynamic processes in such different media as metals and soils. These equations have the form

$$s_{ij}s_{ij} = T(p), \qquad ds_{ij}/dt + \lambda s_{ij} = 2Ge_{ij} \tag{1.1}$$

where

$$s_{ij} = -\sigma_{ij} + p\delta_{ij}, \qquad e_{ij} = \varepsilon_{ij} - {}^1/_3\varepsilon_{ll}\delta_{ij}, \qquad \lambda = (2Ge_{ij}s_{ij} - {}^1/_2 dT/dt)/T$$

Here σ_{ij} and ε_{ij} are tensors of stresses and velocities of deformation, $p = {}^1/_3\sigma_{ii}$ is the pressure, G is the shear modulus, the operator d/dt is an absolute derivative in the sense of [3]. (It is assumed that the summation is carried out over recurring indices $i, j, k = 1, 2, 3$. Compressive stresses are taken as positive.)

The first of equations (1.1) is the plasticity condition of Mises. The function $T(p)$ which enters into this condition is taken in the form $T = 2(kp + b)^2$ where k and b are constants. The particular form of $T(p)$ was selected by us on the basis of mathematical convenience. However, experimental data for the soil [4] give just this type of relationship.

The remaining equations (1.1) express the condition of coaxiality of stress tensors and velocity tensors of plastic deformations. The value of λ is selected such that the condition of plasticity is a consequence of these equations. In this connection it is assumed that $\lambda > 0$. If it turns out that $\lambda < 0$, then (1.1) should be replaced by the conventional equations of elasticity.

The system of equations (1.1) must be closed by means of some relationship between the pressure and the density. This relationship can be quite complex. For example, it can contain hysteresis loops.

So far not a single fairly general solution of equations (1.1) is known. In the solution of specific problems, therefore, these equations are usually simplified. For example, in problems related to the propagation of explosion waves in soils it is frequently possible to distinguish the front of the shock wave and the region of unloading. Then from (1.1) and the relationship between pressure and density it is sometimes possible to derive a simple equation of shock loading. In the region of unloading we can attempt to describe the medium as an ideal fluid or to assume that all quantities depend on one variable only (method of columns), etc.

Such simplification is imperfect (in connection with this case) from a logical point of view, however, sometimes it allows to describe the phenomenon in an acceptable manner.

Now a specially selected boundary value problem will be examined which allows to obtain the exact solution of equations (1.1) together with equations of motion. The same problem is solved approximately by the column method and for the ideal fluid. A comparison is made of solutions obtained.

2. In the xy-plane let the medium occupy the rectangle $l \geqslant y \geqslant 0$, $L \geqslant x \geqslant 0$ (Fig. 1). On the top side of the rectangle the normal stress is given, on the other sides the normal velocity. Tangential stresses at the boundary are equal to zero. The medium is incompressible and is under conditions of plane strain. The defining equations initially are taken in the form (1.1). The density is assumed to be equal to one. Compressive stresses are considered to be positive.

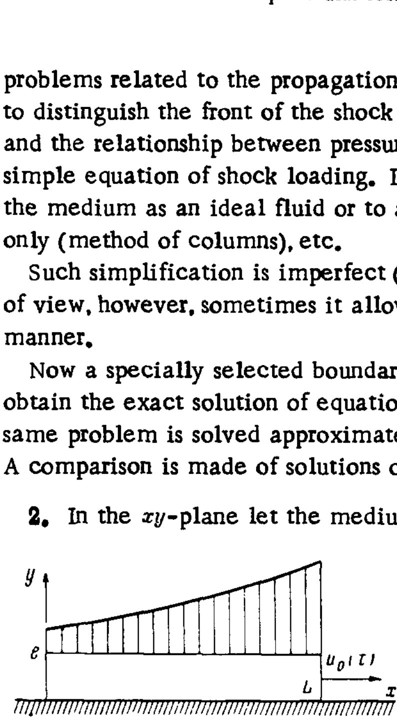

Fig. 1

With these designations the equations of motion and the equations of continuity take the form

$$\frac{du}{dt} = -\frac{\partial \sigma_1}{\partial x} - \frac{\partial \tau}{\partial y}, \quad \frac{dv}{dt} = -\frac{\partial \sigma_2}{\partial y} - \frac{\partial \tau}{\partial x}, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.1}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{u\partial}{\partial x} + \frac{v\partial}{\partial y}$$

Here $\sigma_1 = \sigma_{xx}$, $\sigma_2 = \sigma_{yy}$ and $\tau = \sigma_{xy}$ are stresses, u and v are velocities with respect to x and y.

For an incompressible medium under conditions of plane strain it follows from (1.1) that $p = {}^1/_2\,(\sigma_1 + \sigma_2)$ then (1.1) can be written in the form

$$\frac{d\,(\sigma_1 - \sigma_2)}{dt} + \lambda\,(\sigma_1 - \sigma_2) = 2G\left(-\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{2.2}$$

$$\frac{d\tau}{dt} + \lambda\tau = -G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad (\sigma_1 - \sigma_2)^2 + 4\tau^2 = 4\,(kp + b)^2 \tag{2.3}$$

where λ is the same as in (1.1). Let the boundary condition on the top side of the rectangle be given in the form

$$\sigma_2 = q_0\,(t) + q_1(t)\,x^2 \equiv Q\,(t,\,x) \tag{2.4}$$

on the remaining sides $u = 0$ for $x = 0$, $u = u_0\,(t)$ for $x = L$ and $v = 0$ for $y = 0$. It is assumed also that $q_1\,(t)$ and $u_0\,(t)$ are connected by the following relationship

$$(1 + k)\,(L du_0 \,/\, dt + u_0^2\,(t)) + 2L^2(1 - k)\,q_1\,(t) = 0 \tag{2.5}$$

Initial conditions are taken as

$$\tau = 0, \quad \sigma_1 = \frac{1-k}{1+k}\sigma_2 - \frac{2}{1+k}b, \quad u = u_0(0)\frac{x}{L}, \quad v = -u_0(0)\frac{y}{L}$$

$$\sigma_2 = q_0(0) + q_1(0)x^2 + \frac{l^2 - y^2}{2}\left(\frac{u_0^2(0)}{L^2} - \frac{1}{L}\frac{du_0}{dt}\right) \quad \text{for } t = 0 \qquad (2.6)$$

These conditions contain only two arbitrary functions of time. The dependence on coordinates is fixed.

Such a special selection of initial and boundary conditions is made because in this case the problem has a simple solution which is given by Eqs. (2.5), (2.6) if in these equations q_0 (0), q_1 (0) and u_0 (0) are replaced by q_0 (t), q_1 (t) and u_0 (t). Equations (2.5), (2.6) are simply guessed. The fact that they give a solution of the problem is checked by straight substitution.

In fact, since u does not depend on y, and v on x, while $\tau = 0$, the first of Eqs. (2.3) is satisfied. It is evident from the expression for σ_1 that the second of Eqs. (2.3) is also fulfilled. Equation (2.2) is a corollary of (2.3). Finally, if (2.5), (2.6) are substituted into (2.1), then after some simple calculations we can become convinced that these equations are also satisfied. As far as the condition $\lambda > 0$ is concerned, it acquires the following form, taking into account that $\tau = 0$:

$$4G(\sigma_1 - \sigma_2)\frac{\partial u}{\partial x} + \frac{dT}{dt} < 0 \qquad (2.7)$$

For this condition to be satisfied, it is sufficient to require $u_0 > 0$, $p > 0$ (the material is compressed) and $p_t' < 0$ (unloading).

It is easy to show that if the boundary conditions satisfy the inequalities $q_0(t) > b$, $q_1(t) > 0$, $q_1'(t) < 0$, $q_0'(t) < 0$ and $u_0(t) > 0$, then the condition (2.7) is satisfied. From now on we consider these inequalities as satisfied.

The quantity u_0^2 which enters into (2.5), (2.6) corresponds to convective terms in the equation of motion (2.1). Frequently these terms are completely discarded. The obtained equation allows to estimate the error which arises in connection with this. Since u_0^2 enters into the equation only in the combination $u_0^2 \pm u_0'L$, the convective terms can be neglected if $u_0^2 \ll u_0'L$.

If we set $u(t) = wt$ and $w = \text{const}$, then this condition reduces to $t \ll L / u$, i.e. convective terms can be neglected as long as the displacement of the right boundary is small in comparison to the length of the rectangle. However, this assumption is made when the displacement of the boundary under the action of the load is not taken into account in the boundary conditions.

Neglecting convective terms and making simple transformations, we write the expression for σ_2

$$\sigma_2 = Q(t, x) + \frac{1-k}{1+k}q_1(t)(l^2 - y^2) \qquad (2.8)$$

The structure of the solution is now completely clear.

3. Let us solve the initial problem approximately by the method of columns and let us compare the obtained solution with the exact one. This method is applicable when l / L is small and it consists in neglecting the derivatives with respect to x in the equations. As a result, a one-dimensional problem is obtained into which x enters as a parameter.

The quantities related to the approximate solution will be designated by the index 1. In the given case it follows from the continuity equation and boundary conditions for

$y = 0$ and $x = 0$ that $u^1 = v^1 = 0$. From the second equation (2.1) and the boundary condition for $y = l$ we obtain that $\sigma_2^1 = Q(t, x)$. It is evident from (2.8) that the error is maximum for $y = 0$. After simple transformations the expression for the relative error $\Delta\sigma_2$ assumes the following form (for $y = 0$):

$$\Delta\sigma \equiv \frac{\sigma_2 - \sigma_2^1}{\sigma_2^1} = \frac{1-k}{1+k}\left(\frac{l}{L}\right)^2 \frac{Q(t, L) - Q(t, 0)}{Q(t, x)} \tag{3.1}$$

From here the character of the error is quite clear. It should only be noted that the case $k = 1$ is impossible, because it follows from [5] that $k < 1$.

In Fig. 2 the dependence of σ_2 on x is shown for $y = 0$. Curve *1* is for the exact and curve *2* is for the approximate solution.

4. Let us examine the same problem assuming that the medium is an ideal fluid. Neglecting convective terms, we obtain instead of (2.1)

$$\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0, \qquad \frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} = 0, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4.1}$$

Boundary conditions and initial conditions for velocities remain the same as before. As far as the initial conditions for stresses are concerned, they are not necessary at all because of the properties of an ideal fluid. Since in the initial conditions $u_y - v_x = 0$, the motion is irrotational and it is possible to introduce the potential φ, which satisfies the Laplace equation. The solution will be expressed in terms of φ according to the equations

$$p = -\varphi_t, \qquad u = \varphi_x, \qquad v = \varphi_y, \qquad \Delta\varphi = 0 \tag{4.2}$$

We note that the equations of the ideal fluid are obtained from the Prandtl-Reuss equations if the constants k and b which enter into the plasticity condition are set equal to zero. However, the problem examined in this section is not a special case of the previous one, because in the problem from Sect. 2 the parameter k enters not only into the equations but also into the boundary conditions as a result of the selection of relationship (2.5) for the connection between $q_1(t)$ and $u_0(t)$.

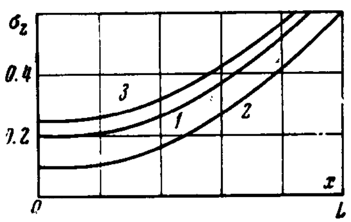

Fig. 2

The boundary value problem for the Laplace equation in the rectangular region is readily solved by the Fourier method. In the given case it is expedient to separate the particular solution φ_1 which satisfies the boundary conditions everywhere with the exception of the right side and also satisfies the initial conditions. For the remaining part of the solution $\varphi_2 = \varphi - \varphi_1$ a boundary value problem is obtained with boundary conditions homogeneous on three sides. This simplifies the matter.

We select φ_1 in the form

$$\varphi_1 = -\int_0^t q_0(s)\,ds + (x^2 + l^2 - y^2)\left(\frac{u_0(0)}{2L} - \int_0^t q_1(s)\,ds\right) \tag{4.3}$$

It is not difficult to see that φ_1 satisfies the Laplace equations and boundary conditions everywhere except at $x = L$. For φ_2 the following boundary value problem is obtained:

$$\varphi_{2x} = 0 \quad \text{for} \quad x = 0, \qquad \varphi_{2y} = 0 \quad \text{for} \quad y = 0, \qquad \varphi_{2l} = 0 \quad \text{for} \quad y = l$$

$$\varphi_{2x} = \frac{4kL}{1+k}\int_0^t q_1(s)\,ds \quad \text{for} \quad x = L \tag{4.4}$$

The solution of this problem has the form

$$\varphi_2 = \frac{4kL}{1+k} f\left(\frac{x}{L}, \frac{y}{l}, \frac{L}{l}\right)\int_0^t q_1(s)\,ds$$

where

$$f(\xi, \eta, r) = \frac{l}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{m^2}\frac{ch\,\pi mr\xi}{sh\,\pi mr}\cos\pi m\eta, \quad m = n + {}^1/_2 \tag{4.5}$$

The function $f(\xi, \eta, r)$ which enters into (4.5) is determined for $0 \leqslant \xi \leqslant 1$, $0 \leqslant \eta \leqslant 1$. For large r the terms of the series which determines the function f have the order of $\exp((\xi - 1)\tau)$. Consequently, the function f differs significantly from zero only for $r \sim 1$. For $L \gg l$ in this manner φ_2 is concentrated near the right boundary.

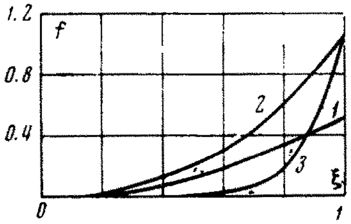

Fig. 3

In Fig. 3 the function $f(\xi, 0, r)$ is shown for $r = 1, 3, 10$ (curves *1*—*3*). If $k = 0$ (the yield limit is independent of pressure), then $\varphi_2 = 0$ and $\varphi = \varphi_1$. In this case the solution assumes a particularly simple form

$$p = Q(t, x) + q_1(t)(l^2 - y^2),$$
$$u = u_0 x / L, \quad v = -u_0 y / L \tag{4.6}$$

A comparison of (4.6) with (2.8) and (2.6) shows that lateral stresses differ by the quantity b. Otherwise the fields of stresses and velocities coincide. If $k \neq 0$, but $L \gg l$, we can set $\varphi = \varphi_1$ everywhere except for $x \sim L$.

In analogy to Sect. 3, we write the expressions for relative errors of approximation of exact solutions by equations of the ideal fluid for $L - x \gg l$

$$\Delta u = \Delta v = -\frac{2k}{1+k} \quad \text{for} \quad 0 \leqslant y \leqslant l$$

$$\Delta\sigma_2 = -\frac{2k}{1+k}\frac{l^2}{L^2}\frac{Q(t, L) - Q(t, 0)}{Q(t, x) + (Q(t, L) - Q(t, 0))\, l^2/L^2} \quad \text{for} \quad y = 0 \tag{4.7}$$

It follows from (4.7) that the assumptions which were made, lead to an increase of velocities (which was apparent in advance) and stresses (curve *3* in Fig. 2). For small k the error has the order $O(k)$. A comparison of (4.7) with (3.1) shows that for the given problem the method of ideal fluid gives a smaller error in modulus than the column method.

In the general case it can be assumed that if k is small and the characteristic stresses are much greater than the yield limit, we can utilize equations of the ideal fluid.

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ON THE INVERSE PROBLEM OF NATURAL VIBRATIONS OF ELASTIC SHELLS

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The problem of determining small changes in the geometric parameters of elastic bodies is considered. It is assumed that the frequency spectrum of their natural vibrations should have given small changes. The method of the small parameter is applied, the problem is reduced to solving an l-moment problem. As an illustration, the problem of determining the variable stiffness of an elastic beam as well as the problem of determining the meridian shape of shells of revolution by means of given frequencies of natural vibrations are considered.

It should be mentioned that the most exhaustive results on such problems exist from the inverse Sturm-Liouville problem [1, 2] as well as for the inverse problem of quantum scattering theory [3, 4].

Only a few papers are devoted to inverse problems of elastic body vibrations. However, the problem of determining the density of an inhomogeneous string by means of its frequency spectra has been investigated with mathematical rigor [5−7]. The problem of determining the stiffness of a beam by means of given natural vibrations frequencies has been considered in an elementary formulation in [8]. This problem has been examined for beams and plates in more detail in [9, 10], where a method is given for the construction of the variable thickness for several given first natural vibrations frequencies and its numerical realization is demonstrated in examples. The present paper is a development of these others.

1. Formulation of the problem. Let us consider the following inverse natural vibrations problem resulting from the first part of [7], under the assumption of smallness in the increments of the natural frequencies.

Let there be the self-adjoint eigenvalue problem

$$Au - \lambda Bu = 0, \qquad G_i u = 0 \qquad (i = 1, \ldots, 2n) \tag{1.1}$$

where

$$Au = \sum_{i=0}^{n} (-1)^i [a_i(\alpha, x) u^{(i)}]^{(i)} \qquad Bu = \sum_{i=0}^{m} (-1)^i [b_i(\alpha, x) u^{(i)}]^{(i)}, \; m < n \tag{1.2}$$